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4D black holes and holomorphic factorization of the 0A matrix model

Ulf H. Danielsson, Niklas Johansson, Magdalena Larfors, Martin E. Olsson and Marcel Vonk

Institutionen för Teoretisk Fysik
Box 803, SE-751 08 Uppsala, Sweden
E-mail: ulf.danielsson@teorfys.uu.se, niklas.johansson@teorfys.uu.se, magdalena.larfors@teorfys.uu.se, martin.olsson@teorfys.uu.se, marcel.vonk@teorfys.uu.se

Abstract: In this paper, we relate the free energy of the 0A matrix model to the sum of topological and anti-topological string amplitudes. For arbitrary integer multiples of the matrix model self-dual radius we describe the geometry on which the corresponding topological string propagates. This geometry is not the one that follows from the usual ground ring analysis, but in a sense its "holomorphic square root". Mixing of terms for different genus in the matrix model free energy yields one-loop terms compatible with type-II strings on compact Calabi-Yau target spaces. As an application, we give an explicit example of how to relate the 0A matrix model free energy to that of a four-dimensional black hole in type-IIB theory, compactified on a compact Calabi-Yau. Variables, Legendre transforms, and large classical terms on both sides match perfectly.

Keywords: Topological Strings, Matrix Models, Black Holes in String Theory.

Contents

1. Introduction

Recently, a very interesting relation between four-dimensional $\mathcal{N}=2$ supersymmetric BPS black holes and topological strings has been proposed [1]. This correspondence relates the black hole free energy in type-IIB string theory, compactified on a Calabi-Yau threefold, to the topological and anti-topological string amplitudes on this same manifold, according to

$$\mathcal{F}_{BH} = \mathcal{F}_{\text{top}} + \overline{\mathcal{F}}_{\text{top}}\,. \tag{1.1}$$

In this relation, the complex structure moduli of the Calabi-Yau are fixed in terms of the black hole charges by certain attractor equations.

Topological strings are often related to matrix models. It is well known that the topological theory on the conifold is perturbatively equivalent to $c=1$ bosonic non-critical string theory at self-dual radius, and hence to a matrix model [2]. In fact, there are quite general correspondences between matrix models and topological strings on non-compact Calabi-Yaus based on ground ring considerations [3–6].

Usually, the advocated correspondence relates the matrix model free energy $\mathcal{F}_{MM}$ directly to the topological string amplitude $\mathcal{F}_{\text{top}}$. However, as is by now well known, the exponential of $\mathcal{F}_{\text{top}}$ should not be viewed as a partition function, but rather as a wave function [7–9]. Thus, it seems unnatural to relate $\mathcal{F}_{MM}$ directly to $\mathcal{F}_{\text{top}}$. If $\mathcal{F}_{MM}$ is a true free energy one should rather make the identification[1]

$$\mathcal{F}_{MM} = \mathcal{F}_{\text{top}} + \overline{\mathcal{F}}_{\text{top}}\,, \tag{1.2}$$

[1]Note that this does not mean that the identification of the $c=1$ free energy with the topological string amplitude is wrong, since in that case one divides the natural (and real) free energy by two so to make it agree with 2d space-time calculations. However, the result is only true for a particular choice of polarization of the phase space $H^3(CY)$.

thereby directly relating the free energy of the matrix model to the one for the black hole. We will see many indications in this paper that this is the right way to think of the relation between matrix models and topological strings. In particular, as we will show in an explicit example in section 3, the black hole free energy resulting from a deformed conifold with complex deformation parameter is precisely given by the free energy of the 0A matrix model at the self-dual radius. From the construction we present in section 2, the generalization to n times the self-dual radius, and its interpretation in terms of n conifolds, is straightforward.[2] The underlying property of the 0A model which makes this interpretation natural is its holomorphic factorization, as we will discuss in section 2.

Note that the identification of the 0A model at self-dual radius with a *single* conifold is different from what one naively may expect from ground ring relations. The reason for this is exactly the holomorphic factorization, which forces us to look at a manifold which in a sense is also the "holomorphic square root" of the one given by the defining relation of the matrix model ground ring. By studying the 0A and 0B matrix models [11, 12] at various radii, and investigating their relation to the $c = 1$ matrix model, we show how these new geometries are constructed.

The procedure also gives new insights on the genus expansions of the matrix model free energy. In particular there will be a crucial mixing between genus 0 and genus 1 terms. The result of the mixing is that the genus 1 term in the expansion of the free energy will have the numerical coefficient $-1/12$ for both the holomorphic and the anti-holomorphic term. As explained in [13], and later pointed out in [2], this is what is required for string theory to resolve the singularity associated to the shrinking of a cycle in the geometry. Again, this result is different from what one may expect from a naive ground ring analysis.

We then turn to the correspondence between matrix models and black holes. As has been argued recently [10], the relations mentioned above allow for a description of the black hole entropy[3] in terms of the free energy of a matrix model. However, there are some important issues that need to be clarified in the proposed correspondence. Most importantly, in [10] non-compact Calabi-Yaus are considered as internal spaces for string compactification. When we view these as local models for compact Calabi-Yaus, we would like to think of all parameters of the noncompact Calabi-Yau as moduli. Hence, the manifold must have at least one A-cycle at infinity, and it is not straightforward to obtain the dependence of the black hole entropy on all the charges. In the present letter we remedy this deficiency by treating compact Calabi-Yaus. It may sound strange that a matrix model can say something about string theory on a compact Calabi-Yau. The reason this happens here is that one can find special charge configurations which, through the attractor equations of [14, 15], result in a singular compactification space only at the horizon. This is enough to allow one to calculate the black hole free energy from the matrix model.

[2] This will also serve to clarify some of the results in [10], where the issue of associating specific radii to a certain number of conifolds was discussed.

[3] There has been some discussion recently on the question whether the quantity calculated in [1] should really be called an entropy, or rather an index. We will use the term "entropy" throughout this paper, but the reader should be aware that this term is not to be taken too literally.

Thus, our internal space is a truly compact manifold which only near the black hole horizon "decompactifies" into a conifold-like geometry. Note that this decompactification only involves quantities that are expressed in terms of the complex structure moduli. In particular, it seems perfectly possible to keep the Kähler volume of the Calabi-Yau finite throughout space-time. Since it is this volume which appears in the four-dimensional Newton's constant, we can really speak of four-dimensional gravity with nonzero coupling constant (and hence, for instance, of true black holes) in this context.

The paper is organized as follows. In section 2 we introduce the matrix models and describe the new geometrical interpretation of the 0A matrix model. We explain the crucial mixing of genus 0 and genus 1 terms that takes place at multiples of the self-dual radius, and its implications for which geometry one should consider. We also work out the case of fractional radii. In section 3 we discuss the correspondence between black holes and matrix models. We describe how the attractor equations can "decompactify" a compact internal space on the black hole horizon. As an explicit example, we study a Calabi-Yau with a conifold point. We match variables, Legendre transforms, and large classical terms on the matrix model and black hole sides. Finally, we summarize and discuss our results.

While this manuscript was being prepared for submission, we received the interesting paper [16], which discusses the 0A and 0B matrix models in a lot of detail. Also in that paper, the holomorphic factorization of the matrix model partition functions plays an important role.

2. The geometry of the 0A matrix model

This section describes the geometrical interpretation we propose for the 0A matrix model. Let us begin by explaining the matrix model nomenclature we use. In the eigenvalue description, the $c = 1$ matrix model describes free fermions in an inverted harmonic oscillator potential, with the Fermi sea filled on one side of the potential. This model is nonperturbatively unstable due to tunnelling. The 0A matrix model[4] differs from this model by a term M/x^2 which is added to the potential. This deformation effectively removes one side of the potential, thus creating a stable model with one Fermi sea. One could also consider the undeformed matrix model with both sides of the potential filled. This corresponds to the 0B matrix model, which also is non-perturbatively stable. The 0A and 0B matrix models were constructed in [11, 12], and their relations at different radii were discussed in detail in [10], to which we refer for further reading.

We will mainly be interested in matrix model free energies. Unless stated otherwise, the free energies are given in the grand canonical ensemble. We use the notation

$$\mathcal{F}_{MM} = -2\pi\beta R F_{MM} = \ln \mathcal{Z}_{MM} \,, \tag{2.1}$$

where MM can stand for "$c = 1$", "0A" or "0B". $\mathcal{Z}_{MM}$ and F_{MM} are the usual partition function and free energy of the matrix model. In the case of the $c = 1$ matrix model, we

[4]The particular model we study was introduced and further studied in [17]-[25], and is also known in the literature as the "deformed matrix model". It can be shown by integrating out eigenvalue phases that the gauged and *holomorphic* matrix model that is often used to describe the two-dimensional type 0A string theory is equivalent to this hermitean matrix model.

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have

$$\mathcal{F}_{c=1}(\mu, R) = \mathrm{Re}\, f(i\mu, R) \tag{2.2}$$

with

$$f(i\mu, R) = \sum_{n,m=0} \ln\left(\frac{2n+1}{2} + \frac{2m+1}{2R} + i\mu\right). \tag{2.3}$$

The genus expansion becomes

$$\mathcal{F}_{c=1}(\mu, R) = -\frac{R}{2}\mu^2 \ln(\mu) - \frac{1}{24}\left(R + \frac{1}{R}\right)\ln(\mu) + \cdots . \tag{2.4}$$

Throughout the paper, we use units where $\alpha' = 1/2$ on the 0A and 0B sides and $\alpha' = 1$ on the $c = 1$ side. This means that the various self-dual radii are $R^A_{SD} = \frac{1}{2}, R^B_{SD} = 1$ and $R^{c=1}_{SD} = 1$, respectively.

The expansion for the 0A free energy [10] is given by

$$\begin{aligned}\mathcal{F}_{0A}(\mu, R, q) &= 2\,\mathrm{Re}\left[f\left(\frac{q+i\mu}{2}, 2R\right)\right] \\ &= 2\,\mathrm{Re}\left[R\left(\frac{q}{2} + \frac{i\mu}{2}\right)^2 \ln\left(\frac{q}{2} + \frac{i\mu}{2}\right) - \right. \\ &\qquad \left. - \frac{1}{24}\left(2R + \frac{1}{2R}\right)\ln\left(\frac{q}{2} + \frac{i\mu}{2}\right) + \ldots\right]\end{aligned} \tag{2.5}$$

where q is related to the coefficient of the deformation term in the potential as $q^2 = M + 1/4$. In the corresponding two-dimensional string theory, q is the net amount of D0-brane charge in the background. The above formula explicitly displays the holomorphic factorization mentioned in the introduction. In a very precise sense, the 0A partition function is the holomorphic square of the "complexified" $c = 1$ partition function.

Let us explain some of the perhaps strange-looking factors of i in the above formulae. In the literature one usually encounters expansions in the parameters μ or $\mu + iq$. However, as emphasized in [10], it is really the sign in front of μ that changes when taking the complex conjugate. This may seem like an academic point since all the signs are going to be squared away anyway. However, it will be important when we construct the geometries, and natural later on when we are matching variables. Moreover, it *will* matter for the nonperturbative part of the theory [10].

2.1 The 0A matrix model at self-dual radius

By going to a double scaling limit, it has been shown that the free energy of the $c = 1$ matrix model at self-dual radius is identical to the topological string amplitude on the conifold [2]. The two terms in eq. (2.4) then correspond to the genus 0 and genus 1 terms of the topological string. From eq. (2.5) we also see that at self-dual radius, the free energy of the 0A matrix model is identical to the sum of the topological and anti-topological amplitudes of the conifold. We have the relation

$$\mathcal{F}_{0A}(\mu, R^A_{SD}, q) = 2\,\mathrm{Re}\left[f\left(\frac{q+i\mu}{2}, R^{c=1}_{SD}\right)\right] = 2\,\mathrm{Re}\left[\mathcal{F}_{\mathrm{top}}\left(\frac{q+i\mu}{2}\right)\right], \tag{2.6}$$

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which holds to all orders in the perturbative genus expansion. Thus, we relate the 0A matrix model at self-dual radius to the conifold, with the equation

$$uv + (\mu - iq) = st \,. \tag{2.7}$$

Note that the 0A theory has enough real parameters to describe one complex modulus. Only in the above expression and similar ones that follow, in order to make contact with existing literature, we use the "conventional" notation where μ is the real part of the parameter and q the imaginary part. Note that we can do this without loss of generality, since we can always for example rescale u and s by a factor of i.

Of course, since we take the real part of f, we could just as well have written it as a function of $q - i\mu$, leading to a conifold of the form

$$uv + (\mu + iq) = st \,. \tag{2.8}$$

Now, we can see how these two manifolds are related to the usual ground ring geometry. It has been proposed [5, 10] that the ground ring equation for the 0A model at self-dual radius including both its deformations is

$$(uv + \mu)^2 = st - q^2 \,. \tag{2.9}$$

We can rewrite this as

$$(uv + \mu - iq)(uv + \mu + iq) = st \,. \tag{2.10}$$

It is useful to view this geometry as a fibration over the uv-plane. The fiber $st = \text{const}$ is a cylinder, except over the loci $uv + (\mu \pm iq) = 0$, where it is the intersection of two complex planes in a single point. As is well-known (see the appendix of [10] for a pictorial explanation) the complex structure moduli of the manifold are related to A-cycles which are localized near these loci, and B-cycles which start there and run off to infinity. Since for q large these loci are far away from each other, the geometry of eq. (2.10) then effectively reduces to two independent copies of the deformed conifolds we mentioned above.

This is the intuitive reason why in the right polarization and perturbatively, the topological amplitude on the ground ring geometry and the sum of topological and anti-topological conifold amplitudes give the same result. Even though in the derivation of (2.6), we have not used the ground ring structure at all, we feel it is the deeper reason for this equation. One may wonder if viewing the defining relation of a Calabi-Yau manifold as a ground ring equation for a two-dimensional string theory, and describing this theory by a matrix model, can lead to an algorithm for calculating the topological string free energy in other cases. We will make some speculative remarks about this in the next subsection.

We would like to stress the words "in the right polarization and perturbatively" in the above paragraph. In fact, it was shown in [6] that the topological B-model on the "doubled conifold" (2.10) has a free energy which to all orders in perturbation theory equals the free energy of the 0A matrix model at self-dual radius. However, to equate a topological string free energy (which is the logarithm of a wave function) to an ordinary free energy, one needs to choose a polarization on the space of three-forms of the manifold on which the topological theory lives. In [6], one works in the so-called holomorphic polarization, and it is only in this polarization that the whole perturbative free energy of the B-model on the doubled conifold can be equated to (2.6).

Since the 0A model is well defined beyond its perturbative expansion [11, 12], it would clearly be interesting to further explore also its nonperturbative aspects at special radii. Because of the above remarks on polarizations, it seems more natural to consider the 0A model free energy as a nonperturbative completion of (2.6) than as a nonperturbative completion of the B-model free energy on the doubled conifold.

2.2 Deformations

The equality of (2.6) — i. e. the sum of the B-model free energy on the conifold and its complex conjugate — with the B-model free energy on (2.10), leads to the following interesting question.[5] If we slightly deform the 0A theory by deforming its ground ring, the corresponding Calabi-Yau manifold (2.10) gets deformed as well. Can such deformations be interpreted as deformations of the two single conifolds in (2.6)? We certainly do not have a complete answer to this question, but in this section we want to make some preliminary remarks.

Let us begin by observing that the B-model on a manifold of the type

$$st = (uv - \mu_1) \cdots (uv - \mu_n)\,, \tag{2.11}$$

which is equipped with the standard holomorphic three-form

$$\Omega = \frac{ds \wedge du \wedge dv}{s}\,, \tag{2.12}$$

has a genus zero free energy given (up to arbitrary normalization) by

$$\mathcal{F}_0 \sim \sum_{i=1}^{n} \mu_i^2 \ln(\mu_i)\,, \tag{2.13}$$

as can be calculated using standard methods — see for example the explanation in [10]. Thus, at least at genus zero it is clear that if the B-model on

$$st = uv - \mu + \epsilon(uv)\,, \tag{2.14}$$

where $\epsilon(uv)$ is a function of which we think as a small deformation, has genus zero free energy $\mathcal{F}_0$, then the B-model on

$$st = (uv - \mu + \epsilon(uv))\,(uv - \mu + \overline{\epsilon}(uv)) \tag{2.15}$$

has genus zero free energy equal to $\mathcal{F}_0 + \overline{\mathcal{F}}_0$.

[5]We thank a referee for pointing this out to us.

Thus, it seems that at least at genus zero and for the above particular class of deformations, there is a relation between deformations of the 0A ground ring and those of the $c = 1$ ground ring. There are however several comments to be made here:

- From the matrix model point of view, the most natural deformations to look at are not deformations in (u, v) but deformations in (s, t). The reason for this is that we obtain the equations for the fermi seas of the 0A and $c = 1$ matrix models by setting $uv = 0$ [10].

- Of course locally, the above equations can be inverted to $(uv - \mu) = st + \tilde{\epsilon}(st)$ and $(uv - \mu)^2 = st + \hat{\epsilon}(st)$. In fact, in this way one arrives precisely at the class of ground ring deformations that was studied first in [3], but of course more general (s,t)-dependent deformations are possible [26, 27]. The inversion amounts to studying only the three-cycles that appear in the original manifolds (2.7) and (2.10), but ignoring new three-cycles (and the corresponding moduli) that come in "from infinity". It is known [28 − 30] that such extra moduli do play an important role in understanding the full topological string theory. However, their size will be of order $\mu_i \sim 1/\epsilon$, and so their contribution in the $1/\mu_i$ expansion will be very small.[6]

- The straightforward factorization we mentioned above requires that we use the standard three-form (2.12). However, from the matrix model point of view, the natural three-form to consider is
$$\Omega = \frac{1}{2\pi i} \frac{du \wedge dp \wedge dx}{u}, \tag{2.16}$$
which reduces to the standard two-form $dp \wedge dx$ on the matrix model phase space. Most likely, this is also the three-form one should use to generalize our statements to higher genus, similarly to [4]. Since the map between (p, x) and (s, t) is nonlinear after deformations, one therefore obtains a three-form which is different from (2.12). One then first has to change coordinates to obtain the standard three-form before the above factorization property can be used. In fact, when we throw away the extra moduli as we have done above, the manifolds we are studying have not changed topology, and by a clever choice of coordinates we can always absorb all deformations into deformations of μ and q only.

Clearly, the above points deserve further investigation, and it would for example be very interesting to see if the factorization also makes sense for more general deformations and at higher genus. All of these issues are currently under investigation by the authors, and we hope to come back to them in a future publication.

2.3 The 0A matrix model at other radii

It is natural to ask whether the 0A matrix model at other radii corresponds to topological string theories on other singular geometries. It was argued in [31, 13] that the type-II string

[6]Except at genus zero and one, where these contributions are in fact infinite in the limit $\epsilon \to 0$ and need to be regulated.

compactified on a compact Calabi-Yau with shrinking cycles is only non-singular if the corresponding topological string amplitude has a one-loop term coefficient of $-k/12$, $k \in \mathbb{Z}$. eq. (2.5) shows that, apart from at self-dual radius, the 0A matrix model does not satisfy this requirement. Hence, it seems that we are in big trouble if we want to identify the general matrix models we consider with topological strings — in particular if we would like these topological strings to live on double scaling limits of compact manifolds, as we will in the next section.

However, because of the double scaling limit the parameters in the model will be of the same order of magnitude as Planck's constant, and it is not immediately obvious anymore that one can match the expressions on the matrix model and the topological string side genus by genus. In particular, it might be the case that the genus 0 term on the matrix model side contributes to the genus 1 term on the topological string side. Similar types of genus mixing have previously been considered in [32, 33]. Below we argue that this is the correct way of viewing the matrix model free energy.

As a motivation we show that, in order for the 0A expression to reduce to 0B as $M \to 0$, we need to make such a reinterpretation of terms. If we plug in $q = \frac{1}{2}$ in eq. (2.5) we get an expression

$$\mathcal{F}_{0A}\left(\mu, R, \frac{1}{2}\right) = 2\,\mathrm{Re}\left[\frac{R}{4}\left(\frac{1}{4} - \mu^2\right)\ln\left(\mu + i\frac{1}{2}\right) - \frac{1}{24}\left(2R + \frac{1}{2R}\right)\ln\left(\mu + i\frac{1}{2}\right) + \cdots\right], \tag{2.17}$$

where we have skipped imaginary and analytic terms. We see that we cannot match the genus 0 (1) term in this expression to the genus 0 (1) term of the 0B free energy directly. For example, the self-dual radius of the 0B model is 1, while in this expression it appears to be 1/2. However, if we move the $\frac{R}{16}\ln(\mu + iq)$ from the genus 0 term to the genus 1 term in eq. (2.17), we indeed get the expression for half the 0B free energy.[7] Note that we also get the correct self-dual radius for 0B by making this shift.

We now turn to the reinterpretation of terms suitable for describing topological strings on Calabi-Yau manifolds. To this end we use the formula of Gopakumar and Vafa [34] for $\mathcal{F}_{c=1}(R_{SD}^{c=1}/n)$:

$$\mathcal{F}_{c=1}\left(\mu, \frac{R_{SD}^{c=1}}{n}\right) = \sum_{k=-(n-1)/2}^{(n-1)/2} \mathcal{F}_{c=1}\left(\frac{\mu - ik}{n}, R_{SD}^{c=1}\right). \tag{2.18}$$

Using eq. (2.5) and going to n times the self-dual radius, this can be recast into a formula for $\mathcal{F}_{0A}$:

$$\mathcal{F}_{0A}(\mu, nR_{SD}^{A}, q) = 2\,\mathrm{Re}\left[\sum_{k=-(n-1)/2}^{(n-1)/2} f\left(\frac{(q + \frac{2k}{n} + i\mu)}{2}, R_{SD}^{c=1}\right)\right]. \tag{2.19}$$

The 0A free energy at n times the self-dual radius is thus given as two times the real part of the sum of n $c = 1$ free energies at self-dual radius. Since the coefficient in front of the 1-loop term of each f is $-1/12$, this immediately shows that we have succeeded in

[7] We only get half since in the 0A theory, half of the states have to be removed [10].

rearranging the terms so that they make sense from a type-II string theory point of view. It also means, by the result of Ghoshal and Vafa [2], that it computes a sum of $2\,\mathrm{Re}\,\mathcal{F}_{\mathrm{top}}$ on n conifolds.

We are now in a position to say something about the geometrical interpretation of the 0A matrix model at n times the self-dual radius. It should correspond to a certain double scaling limit of the topological theory on a Calabi-Yau with n three-cycles that can shrink at different loci in moduli space. Call the distance to these loci t^k, $k = -\frac{n-1}{2}, \ldots, \frac{n-1}{2}$. Then the limit described by the 0A matrix model is $t^k \to 0$ and $g_{\mathrm{top}} \to 0$ with

$$\frac{t^k}{g_{\mathrm{top}}} = \left(q + \frac{2k}{n}\right) + i\mu \tag{2.20}$$

kept fixed.[8] Thus the Calabi-Yau must allow for all cycles to shrink simultaneously. Note also that the matrix model, having only two parameters, describes a very special limit of this geometry.

There are of course many geometries satisfying these properties, but some have a more natural interpretation than others. As an example, let us work out the geometry in more detail for the case $n = 2$. At twice the self-dual radius, the free energy is (recall that $R_{SD}^{c=1} = 2R_{SD}^{A} = 1$)

$$\begin{aligned}\mathcal{F}_{0A}(\mu, 1, q) &= 2\,\mathrm{Re}\left[f\left(\frac{q-\frac{1}{2}+i\mu}{2}, 1\right) + f\left(\frac{q+\frac{1}{2}+i\mu}{2}, 1\right)\right] \\ &= 2\,\mathrm{Re}\left[\mathcal{F}_{\mathrm{top}}\left(\frac{q-\frac{1}{2}+i\mu}{2}\right) + \mathcal{F}_{\mathrm{top}}\left(\frac{q+\frac{1}{2}+i\mu}{2}\right)\right].\end{aligned} \tag{2.21}$$

We see that there are two loci in parameter space where the corresponding Calabi-Yau should have conifold singularities. Note that, again, this is half the number one would expect by a naive ground ring analysis. Following the arguments in [34], such a manifold can be created by modding out the conifold $st = uv + (\mu - iq)$ by a $\mathbb{Z}_2$, changing variables from (s^2, t^2) to (s, t), and deforming the resulting A_2-singularity by $\pm\frac{1}{2}$, leading to

$$\begin{aligned}st &= \left(uv + \mu - i\left(q - \frac{1}{2}\right)\right)\left(uv + \mu - i\left(q + \frac{1}{2}\right)\right) \\ &= (uv + \mu)^2 - 2iq(uv + \mu) - M\,.\end{aligned} \tag{2.22}$$

Just as at the end of the previous section, this procedure boils down to simply multiplying the equations for the loci of the single conifolds.

However, perturbatively we can rewrite eq. (2.21) in several other ways, such as

$$\begin{aligned}\mathcal{F}_{0A}(\mu, 1, q) &= 2\,\mathrm{Re}\left[\mathcal{F}_{\mathrm{top}}\left(\frac{q-\frac{1}{2}+i\mu}{2}\right) + \mathcal{F}_{\mathrm{top}}\left(\frac{q+\frac{1}{2}-i\mu}{2}\right)\right] \\ &= 2\,\mathrm{Re}\left[\mathcal{F}_{\mathrm{top}}\left(\frac{q-\frac{1}{2}+i\mu}{2}\right) + \mathcal{F}_{\mathrm{top}}\left(\frac{-\left(q+\frac{1}{2}\right)+i\mu}{2}\right)\right] + \text{non-pert.}\end{aligned} \tag{2.23}$$

[8] Eq. (2.20) is the correct equation if the parameters t^k are chosen so that all $(t^k)^2 \ln t^k$ terms in $\mathcal{F}_{\mathrm{top}}$ appear with the same coefficient, and up to an overall normalization.

where in the first expression we have used $\mathrm{Re}[f(z)] = \mathrm{Re}[f(\bar{z})]$. The second expression can be easily verified by examining the genus expansion. The detailed form of the non-perturbative contribution was given in [10] for the case of $q = 0$. We might then conclude that the resulting manifold is given by

$$\begin{aligned} st &= \left(uv + \mu - i\left(q - \frac{1}{2}\right)\right)\left(uv + \mu + i\left(q + \frac{1}{2}\right)\right) \\ &= (uv+\mu)^2 + M + i\,(uv+\mu)\,. \end{aligned} \tag{2.24}$$

We claim that this latter manifold is the more natural one corresponding to the ground state of the Fermi sea. (By general ground ring arguments [3], one can argue that extra terms proportional to uv and 1 should correspond to excitations of the Fermi sea.) The reason for this is that we can now make contact with the higher genus analysis described in [21], which is equivalent to the Kodaira-Spencer description of the topological string [35]. See also [4, 6], where similar techniques are used. To do this we need to consider [10, 21] a superpotential given by

$$W = \frac{M}{D^2} + \left(\frac{\mu}{D} - X + \sum t_{2k} D^{2k-1}\right)^2, \tag{2.25}$$

where $[D, X] = -i$. For the free energy we need not consider the perturbations and we can put all $t_{2k} = 0$. Hence, commuting everything to the right, and putting $X = 0$, we find

$$W = \frac{M}{D^2} + \frac{\mu^2}{D^2} - i\frac{\mu}{D^2}\,. \tag{2.26}$$

We see that this matches the structure of the expression in eq. (2.24) for $uv = 0$ and up to an irrelevant complex conjugation. For further details on how to perform the higher genus calculations in this framework, see [21]. It would of course be very interesting to see if the manifolds that are natural from the Kodaira-Spencer point of view also allow for a more natural embedding into truly compact Calabi-Yaus.

For completeness, let us work out the case for fractional radii $R^A = R^A_{SD}/n$. In this case, the 0A free energy should be written [34]

$$\begin{aligned} \mathcal{F}_{0A}\left(\mu, \frac{R^A_{SD}}{n}, q\right) &= 2\,\mathrm{Re}\left[f\left(\frac{q+i\mu}{2}, \frac{R^{c=1}_{SD}}{n}\right)\right] \\ &= 2\,\mathrm{Re}\left[\sum_{k=-(n-1)/2}^{(n-1)/2} f\left(\frac{q+2k+i\mu}{2n}, R^{c=1}_{SD}\right)\right]. \end{aligned} \tag{2.27}$$

In terms of the Calabi-Yau, a natural interpretation of this sum, which contains n terms, is that the total charge q and potential μ is associated to the n conifolds in a specific way.[9]

[9] Running slightly ahead of the black hole part of our story, let us make the following interesting observation. For the l'th conifold, say, the attractor equations [14, 15], fix the complex structure moduli (including the imaginary part [1]) at the horizon to

$$CX^{l+1} = \frac{1}{n}(2l + 1 + q + i\mu) - 1\,, \tag{2.28}$$

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3. Black hole entropy and compact Calabi-Yaus

We now turn to the relation between matrix models and black hole entropy. In ref. [10], the relation $S_{BH} = -F_{MM}/T_{MM}$ is derived. F_{MM} is in the canonical ensemble and $T_{MM} = (2\pi R_{MM})^{-1}$ is the matrix model temperature.

Matrix models are usually related to topological strings on non-compact Calabi-Yaus. In the case of a *compact* Calabi-Yau the number of independent three-cycles is $b^3 = 2(h^{(2,1)} + 1)$, and these are naturally divided into symplectic pairs of A- and B-cycles. The complex structure moduli space then has dimension $h^{(2,1)}$. It can be parameterized by the periods of the holomorphic $(3,0)$-form on $h^{2,1}$ A-cycles, or more invariantly by considering the periods on all $(h^{2,1}+1)$ A-cycles as projective coordinates. For more details, the reader is referred to [37] for a review on the special geometry of Calabi-Yaus.

We would like to think of the *non-compact* manifolds as local models for compact Calabi-Yaus, and of the periods of the n A-cycles in these geometries as n true complex structure moduli. This means there has to be at least one extra A-cycle "at infinity". In the formula $S_{BH} = -F_{MM}/T_{MM}$, the left hand side is a function of the black hole charges. Since the number of electromagnetic charges of the four-dimensional black hole equals the total number of three-cycles, it is important to take the extra A-cycle(s) into account when obtaining the dependence of the black hole entropy on the charges in this framework.

In this section we derive an explicit correspondence between the 0A matrix model at self-dual radius and a four-dimensional half-BPS $\mathcal{N} = 2$ black hole on a compact internal space with a conifold point. This involves finding charge configurations that, through the attractor equations [14, 15], fix the moduli to a conifold point at the horizon. Let us however stress again that this "decompactification" only takes place near the black hole horizon, and only for quantities that are sensitive to the complex structure moduli — the Newton's constant being the most notable exception. We will identify variables, thermodynamical ensembles and double scaling limits on both sides, thus tying up one end left loose in ref. [10]. We also verify that the Legendre transforms, taking us from $\mathcal{F}_{BH}$ to S_{BH} on the black hole side [1] and from the grand canonical to the canonical ensemble on the matrix model side, coincide as proposed in ref. [10]. Finally, we explore the large classical contributions present on both sides.

For a review on compactification in the black hole context, see ref. [38]. In [10, 39], two different ways to deal with the truly noncompact case by introducing cutoffs were discussed.

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where $0 \leq l \leq n-1$. It is interesting to compare this with the energy eigenvalues of the 0A matrix model [18]:

$$E^l = i(2l + 1 + q + i\mu)\,. \tag{2.29}$$

Given our identifications, and the fact that the topological partition function is peaked at the attractor value [36], a relation between the attractor fixed point values and the energy eigenvalues of the 0A matrix model is not unexpected.

3.1 Charges and decompactification

Consider for simplicity a Calabi-Yau $\mathcal{M}$ with just one complex structure modulus. For example, one could think of $\mathcal{M}$ as the mirror quintic, which has been thoroughly studied in [40]. Extending the treatment to the general case is straightforward. We choose a symplectic basis A^I, B_I, $I = 0, 1$ of $H_3(\mathcal{M}, \mathbb{Z})$, which is such that the period of A^1 shrinks to zero at the conifold point. Let X^I and F_I be the periods of the holomorphic three-form on A^I and B_I. Each pair of cycles leads to a four-dimensional gauge field, and hence to an electric and a magnetic charge. Our objective is to express the entropy of the black hole as a function of its electromagnetic charges q_I and p^I.

Recall that the entropy is given by [1]

$$S_{BH}(q_I, p^I) = \mathcal{F}_{BH}(\phi^I, p^I) - \phi^J \frac{\partial}{\partial \phi^J} \mathcal{F}_{BH}(\phi^I, p^I)\,, \tag{3.1}$$

where ϕ^I are the chemical potentials conjugate to q_I. $\mathcal{F}_{BH}(\phi^I, p^I)$ can be obtained from the topological partition function $\mathcal{F}_{\rm top}$ on the Calabi-Yau as [1]

$$\mathcal{F}_{BH}(\phi^I, p^I) = 2\,\mathrm{Re}\,\mathcal{F}_{\rm top}(t, g_{\rm top})\,. \tag{3.2}$$

Here $t = X^1/X^0$ is a parameter on moduli space, and $g_{\rm top}$ is the topological string coupling constant. The correspondence holds if $\mathcal{F}_{\rm top}$ is evaluated at

$$t = \frac{p^1 + i\phi^1/\pi}{p^0 + i\phi^0/\pi}\,, \qquad g_{\rm top} = \frac{\pm 4\pi i}{p^0 + i\phi^0/\pi}\,. \tag{3.3}$$

We now want to compute $\mathcal{F}_{BH}$ using matrix model technology. To this end we use eq. (2.5), and the fact that $\mathcal{F}_{c=1}$ equals the topological partition function on the conifold [2]. To be more specific, in the double scaling limit $t \to 0$, $g_{\rm top} \to 0$ at constant $\mu_{\rm top} \equiv t/g_{\rm top}$, the partition function is given by

$$\mathcal{F}_{c=1}(\mu = \mu_{\rm top}) = \mathrm{Re}\,\mathcal{F}_{\rm top}(i\mu_{\rm top})\,. \tag{3.4}$$

The equality (3.4) is valid only after appropriately fixing the gauge on the topological string theory side, and up to large classical terms, on which we will comment in a moment. Thus, eqs. (2.5), (3.2) and (3.4) give $\mathcal{F}_{BH} = \mathcal{F}_{0A}$, upon identifying variables. This is done by computing the $\sim (q + i\mu)^2 \ln[(q + i\mu)/\beta]$ contribution to the zero order term of $\mathcal{F}_{0A}$, and its counterpart $\sim (p^1 + i\phi^1/\pi)^2 \ln[(p^1 + i\phi^1/\pi)/(p^0 + i\phi^0/\pi)]$ in $\mathcal{F}_{BH}$. Doing this carefully, using e.g. eq. (2.16) of ref. [1], yields the identification

$$\begin{aligned} \mu &\equiv \frac{\phi^1}{\pi}\,, \\ q &\equiv p^1\,, \\ \beta &\equiv p^0 + \frac{i\phi^0}{\pi}\,. \end{aligned} \tag{3.5}$$

Note that β is to be considered as an independent variable. It appears only in the the genus 0 and 1 contributions, exactly as $p^0 + i\phi^0/\pi$. Let us stress that μ is identified with ϕ^1, and not with p^1.

With the correspondence (3.5) we have $\mathcal{F}_{BH} = \mathcal{F}_{0A}$, and the fact that $\mu \sim \phi^1$ shows that the Legendre transforms on both sides indeed match.[10] Thus we have explicitly verified the conclusion that $S_{BH} = -F_{0A}/T_{SD}$ (eq. (1.2) of ref. [10]), where F_{0A} is the canonical free energy of the 0A matrix model, and $T_{SD} = 1/\pi$ is the self-dual temperature. To be precise, in this ensemble the identification of q and β is as in (3.5), and instead of μ we now have N, which is the number of fermions measured from the top of the $-x^2$ part of the potential. This variable is to be identified with the electric charge q_1 of the black hole, and has expectation value $\langle N \rangle = -\frac{1}{\pi}\partial\mathcal{F}_{0A}/\partial\mu$. Having made the connection (3.5) we need to identify the double scaling limit on the black hole side. Indeed, eq. (3.4) is only valid in that limit, and thus it is only in this limit that $\mathcal{F}_{0A}$ correctly computes the entropy. Eq. (3.3) shows that the appropriate limit is $p^0 + i\phi^0/\pi \to \infty$ while $p^1 + i\phi^1/\pi$ remains constant.[11] Hence p^1 remains constant, and using the attractor equations, it is straightforward to show that at least two of p^0, q_0 and q_1 go to infinity. The attractor equations also give the following condition on these three charges:

$$p^0\,\mathrm{Im}(F_0\bar{F}_1) + q_0\,\mathrm{Im}(F_1\bar{X}^0) + q_1\,\mathrm{Im}(X^0\bar{F}_0) = 0\,, \tag{3.6}$$

where all periods are evaluated at $t = 0$. When the charges satisfy these requirements, the Calabi-Yau will be effectively non-compact at the black hole horizon.

3.2 Classical terms

Next, let us consider the large classical terms appearing in the black hole entropy. Computing the genus 0 contribution gives

$$\mathcal{F}_{BH} = 2\,\mathrm{Re}\left[\frac{\pi i}{4}A_1(CX^0)^2 + \frac{\pi i}{2}A_2(CX^0)(CX^1) + \frac{1}{2}(\frac{CX^1}{2})^2\ln\frac{CX^1}{CX^0} + \cdots\right], \tag{3.7}$$

where terms that are vanishing or finite and regular in the double scaling limit have been omitted. Here, $CX^I \equiv p^I + i\phi^I/\pi$, and A_i are numerical constants depending on the Calabi-Yau. Explicitly $A_1 = (F_0/X^0)|_{t=0}$ and $A_2 = (F_1/X^0)|_{t=0}$. Note that the first two terms become large in the double scaling limit.

In principle, there are large classical contributions also to the matrix model free energy. Regularizing the potential of the $c = 1$ matrix model $V \sim -x^2$ according to

$$V(x) = -\frac{x^2}{\alpha'} + Ax^4\,, \tag{3.8}$$

gives, up to numerical factors, a grand canonical matrix model free energy of the form

$$\mathcal{F}_{c=1}(\mu,\beta) \sim \frac{1}{A^2}\beta^2 - \frac{1}{A}\mu\beta - \mu^2\ln\frac{\mu}{\beta} + \cdots\,. \tag{3.9}$$

[10] On the black hole side the transform really contains a $-\phi^0\partial\mathcal{F}_{BH}/\partial\phi^0$ term which is not present on the matrix model side. Since β only appears in the genus 0 and 1 terms, this contribution only contains non-universal terms. However, since we will be interested also in the non-universal terms, we choose the black hole charges in such a way that $\phi^0 = 0$ in what follows.

[11] Note that this limit coincides exactly with the usual matrix model double scaling limit [41]-[44].

We see that since we can identify $CX^0 \sim \beta$, $CX^1 \sim i\mu$, the structure of this expression is in complete accordance with (3.7). For the example of the mirror quintic, we also checked that the sign of the leading term is the same in both equations. To precisely match the two undetermined coefficients in (3.7), one would need to consider a potential which is regularized by two coefficients, such as $V \sim -x^2 + Ax^4 + Bx^6$.

4. Conclusions and outlook

In this paper, we have studied the correspondence between matrix models, topological strings and four-dimensional $\mathcal{N} = 2$ half-BPS black holes. We have in particular studied the relations between these systems for the case of the 0A matrix model at multiples of its self-dual radius. When relating the matrix models to topological strings, we have argued that it is more natural to match the matrix model free energy to $2\,\mathrm{Re}\,\mathcal{F}_{\mathrm{top}}$ than to $\mathcal{F}_{\mathrm{top}}$. Consequently, the geometry naturally associated to the matrix model at self-dual radius is the deformed conifold. This conifold can be viewed as the "holomorphic square root" of the manifold that follows from the ground ring equations. At multiples of the self-dual radius we again find such a holomorphic–anti-holomorphic factorization, and the matrix model should be associated with certain non-compact Calabi-Yau manifolds with n three-cycles that can shrink to zero volume. We found that it is plausible that such local geometries can be embedded in compact Calabi-Yaus.

We noted that the matrix model free energies and the topological partition functions need not match each other genus by genus. In particular, a mixing of genus 0 and genus 1 terms will occur. This ensures that the coefficients in front of the genus 1 terms on the topological string side are always $-1/12$ for both the holomorphic and the anti-holomorphic contributions, as required for the resolution of compact singular spaces by string theory.

Using the recently conjectured correspondence between topological strings and black holes in type-IIB string compactification, we were able to directly relate the matrix model free energy to the one for the black hole. An important new point here was that the theory is compactified on a compact Calabi-Yau, that through the attractor equations develops a singularity only at the black hole horizon. This allowed us to get a matrix model description of the black hole, in spite of the compactness of the Calabi-Yau.

The relation was calculated explicitly for an internal space with a single conifold point. The variables matched perfectly and we saw that the Legendre transforms between the canonical and grand canonical ensemble on the matrix model side, and between S_{BH} and $\mathcal{F}_{BH}$ on the black hole side were identical. It was also shown that the large classical terms in $\mathcal{F}_{BH}$ and $\mathcal{F}_{MM}$ can be matched in form by regulating the matrix model potential.

There has been much interest in the consequences of the topological string / black hole relation recently, and it seems that many more interesting results in this direction lie ahead. Let us mention some lines of further investigation related to the results of this paper. First of all, it would be extremely interesting to understand the nonperturbative corrections on the different sides of the story better. Many of the models that we have mentioned are perturbatively equivalent, but have nonperturbative differences. Studying these better, as well as their relations to black holes, may give us some intuition about the correct nonperturbative completion of topological string theory, and about the question of how unique such a completion is. A closer nonperturbative study may also lead to relations with the baby universes of [45]. In this respect, the sums over different conifolds we have mentioned are also suggestive.

On a more technical level, the notion of the "holomorphic square root" of the ground ring geometry needs to be made more precise. On a case-by-case basis, the correct geometries are not hard to guess, but it seems that by using Kodaira-Spencer theory a more rigorous definition should also be possible.

Another interesting point to work out further would be the actual embedding of the local models into compact Calabi-Yaus. Ultimately, this leads to the intriguing mathematical question of which local Calabi-Yau manifolds allow an embedding into compact Calabi-Yaus. Already in the two-moduli case this seems to be a very nontrivial issue.

Finally, we repeat an open question that was mentioned in [10]: could we completely skip the topological string step and directly relate the matrix models to the black holes? Since the black holes have an $AdS_2 \times S^2$ near-horizon region, one would expect the supergravity theory to be equivalent to a CFT_1 on the boundary of AdS_2. It seems natural to relate the two factors of the matrix model partition function to the two boundaries of this space. It would be interesting to make such a holographic description precise.

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